Simulation Models in Industrial Engineering

Spring Semester, 2022 - 2023

Midterm Review

Outline

Lecture 1: Introduction to Simulation

Lecture 2: Simulation Examples

Lecture 3: General Principles of Simulation Modelling

Lecture 4: Review Statistics

Lecture 5: Random Numbers

Lecture 6: Random Variate

Lecture 1

Introduction to Simulation

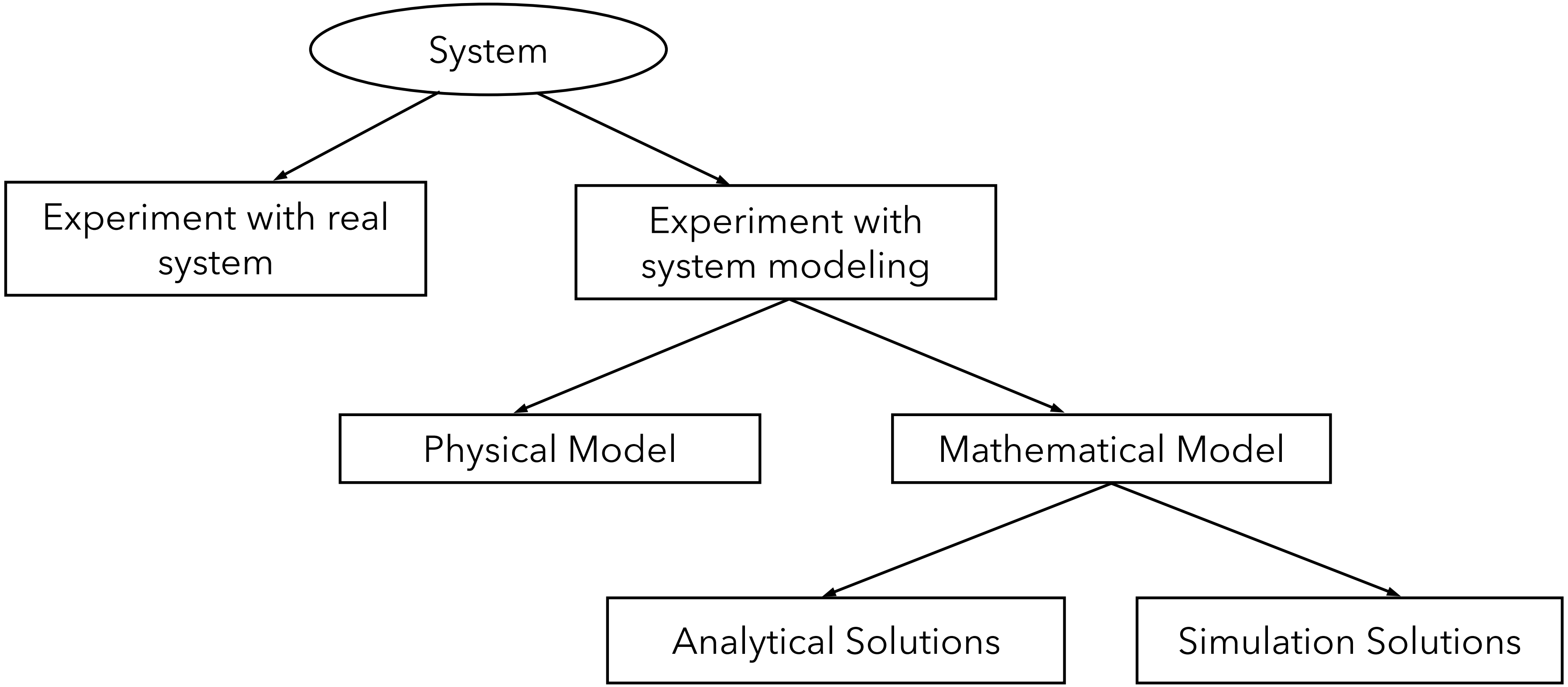
Introduction to Simulation

Review questions:

1. What is a Simulation?
2. When to and NOT to use Simulation?
3. What are advantages and disadvantages of Simulation?
4. What are some areas for applications of Simulation?
5. What are systems, system environment, and components of a system? Example?
6. Distinguish discrete and continuous systems! Example?
7. What are models and some model types of a system?
8. **What are some methods to study systems?**
9. **What are steps in a simulation study?**

Introduction to Simulation

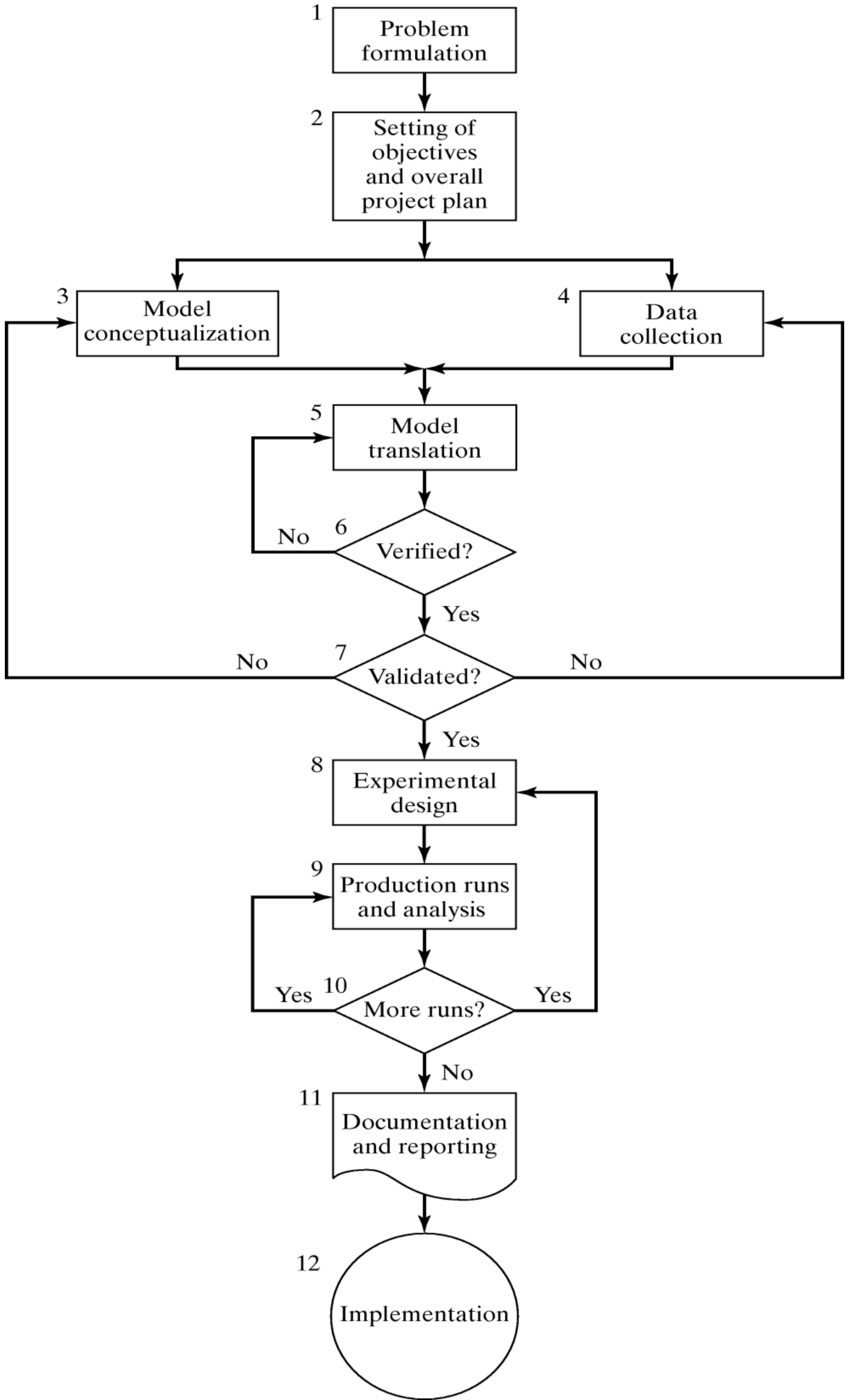
8. What are some methods to study systems?



Introduction to Simulation

9. What are steps in a simulation study?

- Four phases:
  - Problem formulation, setting objective and overall design (step 1 to 2).
  - Modeling building and data collection (step 3 to 7)
  - Running of the model (step 8 to 10).
  - Implementation (step 11 to 12).
- An iterative process.



Bonus questions!

What is the most important step? Why?

Lecture 2

Simulation Examples

Simulation Examples

Simulation of queueing system

Single-channel queue

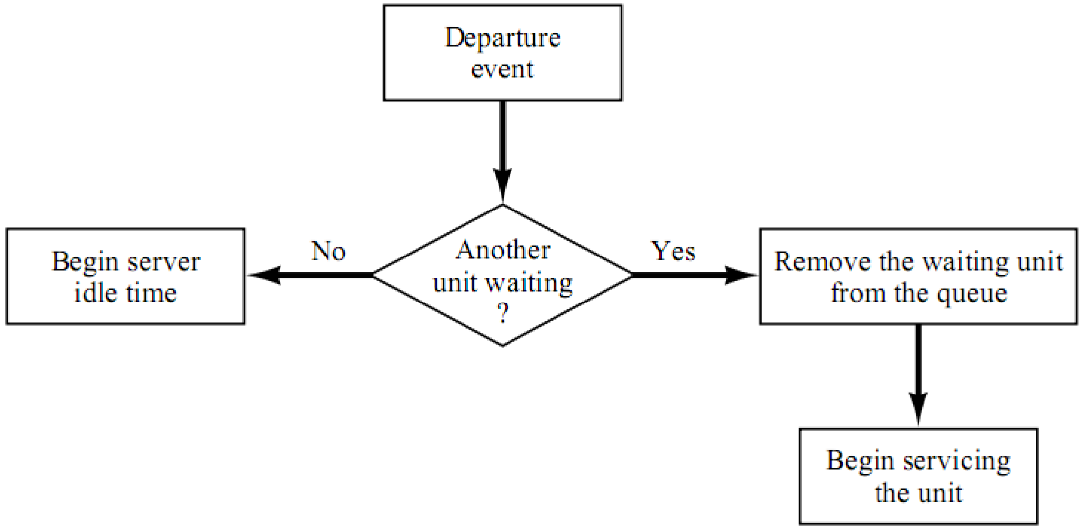


Figure 2.2 Service-just-completed flow diagram.

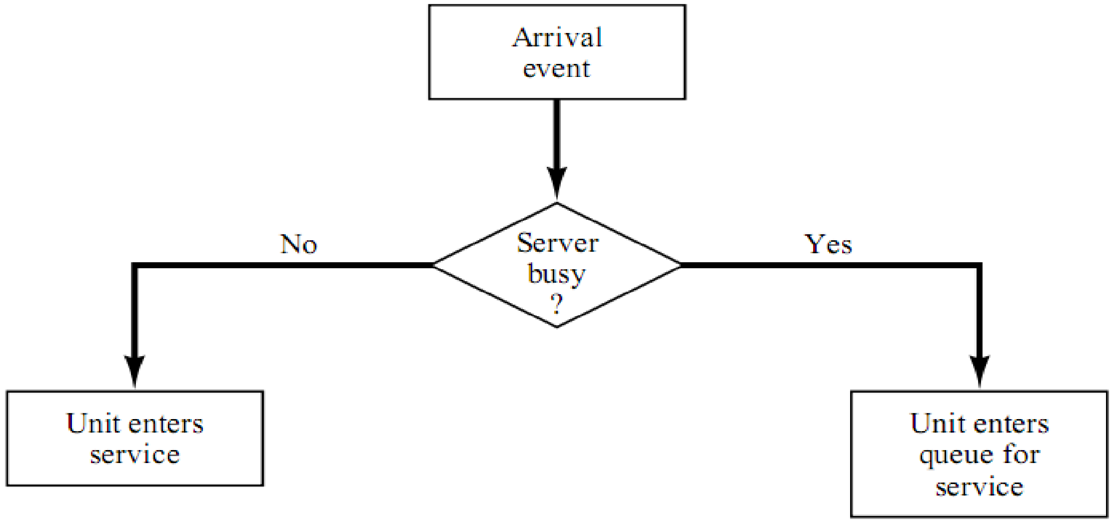


Figure 2.3 Unit-entering-system flow diagram.

Simulation Examples

Simulation of queueing system

Single-channel queue

Table 2.10 Simulation Table for Queueing Problem

A *Customer*	B *Time Since Last Arrival (Minutes)*	C *Arrival Time*	D *Service Time (Minutes)*	E *Time Service Begins*	F *Time Customer Waits in Queue (Minutes)*	G *Time Service Ends*	H *Time Customer Spends in System (Minutes)*	I *Idle Time of Server (Minutes)*
1	–	0	4	0	0	4	4	0
2	8	8	1	8	0	9	1	4
3	6	14	4	14	0	18	4	5
4	1	15	3	18	3	21	6	0
5	8	23	2	23	0	25	2	2
6	3	26	4	26	0	30	4	1
7	8	34	5	34	0	39	5	4
8	7	41	4	41	0	45	4	2
9	2	43	5	45	2	50	7	0
10	3	46	3	50	4	53	7	0
11	1	47	3	53	6	56	9	0
12	1	48	5	56	8	61	13	0
13	5	53	4	61	8	65	12	0
14	6	59	1	65	6	66	7	0
15	3	62	5	66	4	71	9	0
16	8	70	4	71	1	75	5	0
17	1	71	3	75	4	78	7	0
18	2	73	3	78	5	81	8	0
19	4	77	2	81	4	83	6	0
20	5	82	3	83	1	86	4	0
			68		56		124	18

Simulation Examples

Simulation of queueing system

Single-channel queue

$$\text{Average waiting time} = \frac{\text{total time customers wait in queue (minutes)}}{\text{total numbers of customers}} = \frac{56}{20} = 2.8\text{minutes}$$

$$\text{Probability (wait)} = \frac{\text{number of customers who wait}}{\text{total numbers of customers}} = \frac{13}{20} = 0.65$$

$$\text{Probability of Idle Server} = \frac{\text{Total idle time of server}}{\text{total run time of simulation}} = \frac{18}{86} = 0.21$$

$$\text{Average Service Time} = \frac{\text{Total Service Time}}{\text{Total Number of customers}} = \frac{68}{20} = 3.4\text{minutes}$$

$$\text{Average Time between arrivals} = \frac{\text{sum of all times between arrivals}}{\text{number of arrivals - 1}} = \frac{82}{19} = 4.3\text{minutes}$$

$$\text{Average waiting time of those who wait} = \frac{\text{total time customers wait in queue}}{\text{total number of customers that wait}} = \frac{56}{13} = 4.3\text{minutes}$$

$$\text{Average time customer spends in the system} = \frac{\text{total time customers spend in the system}}{\text{total number of customers}} = \frac{124}{20} = 6.2\text{minutes}$$

Or

$$\text{Average time customer spends in the system} = \text{average waiting time} + \text{average service time} = 2.8 + 3.4 = 6.2 \text{ minutes}$$

Simulation Examples

Simulation of queueing system

- Single-channel queue
- Simulation of a two-server queuing system

Simulation of inventory system

Other examples of simulation

General Principles of Simulation Modelling

Major concepts in simulation

System A collection of entities (e.g., people and machines) that interact together over time to accomplish one or more goals.

Model An abstract representation of a system, usually containing structural, logical, or mathematical relationships that describe a system in terms of state, entities and their attributes, sets, processes, events, activities, and delays.

System state A collection of variables that contain all the information necessary to describe the system at any time.

Entity Any object or component in the system that requires explicit representation in the model (e.g., a server, a customer, a machine).

Attributes The properties of a given entity (e.g., the priority of a waiting customer, the routing of a job through a job shop).

List A collection of (permanently or temporarily) associated entities, ordered in some logical fashion (such as all customers currently in a waiting line, ordered by "first come, first served," or by priority).

Event An instantaneous occurrence that changes the state of a system (such as an arrival of a new customer).

Event notice A record of an event to occur at the current or some future time, along with any associated data necessary to execute the event; at a minimum, the record includes the event type and the event time.

Event list A list of event notices for future events, ordered by time of occurrence; also known as the future event list (FEL).

Activity A duration of time of specified length (e.g., a service time or interarrival time), which is known when it begins (although it may be defined in terms of a statistical distribution).

Delay A duration of time of unspecified indefinite length, which is not known until it ends (e.g., a customer's delay in a last-in–first-out waiting line which, when it begins, depends on future arrivals).

Clock A variable representing simulated time, called CLOCK in the examples to follow.

General Principles of Simulation Modelling

World views for developing a model

Event-scheduling world view

- Allow us to control everything; have complete flexibility; know the state of everything anytime
- Easily to be coded up in any programming language or with macros in a spreadsheet
- Become very complicated for large models with lots of different kinds of events, entities and resources

Process-interaction world view

- Focus on the processes that entities undergo
- Analogous to flowcharting
- Employed when using a process-oriented simulation language or a simulation software (e.g., ARENA)
- Most discrete-event simulation are executed in the event orientation even though we cannot see it.

Review Statistics

Hypothesis testing

For population mean (large sample)

	Two-tailed Test	One-tailed Test
Hypothesis	$H_0: \mu = \mu_0$ $H_1: \mu \neq \mu_0$	$H_0: \mu > \mu_0 \; (\mu < \mu_0)$ $H_1: \mu \leq \mu_0 \; (\mu \geq \mu_0)$
Test Statistic	$z_t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$	
Critical points	$\pm z_{\alpha/2}$	$-z_\alpha \; (z_\alpha)$
Decision Rules	Reject H_0 if $z_t > z_{\alpha/2}$ or $z_t < -z_{\alpha/2}$	Reject H_0 if $z_t < -z_\alpha \; (z_t > z_\alpha)$

Review Statistics

Hypothesis testing

For population mean (small sample), s unknown, population is normally distributed

	Two-tailed Test	One-tailed Test
Hypothesis	$H_0: \mu = \mu_0$ $H_1: \mu \neq \mu_0$	$H_0: \mu > \mu_0$ ($\mu < \mu_0$) $H_1: \mu \leq \mu_0$ ($\mu \geq \mu_0$)
Test Statistic	$t_t = \dfrac{\bar{x} - \mu_0}{s / \sqrt{n}}$	
Critical points	$\pm t_{n-1, \alpha/2}$	$-t_{n-1, \alpha}$ ($t_{n-1, \alpha}$)
Decision Rules	Reject H_0 if $t_t > t_{n-1, \alpha/2}$ or $t_t < -t_{n-1, \alpha/2}$	Reject H_0 if $t_t < -t_{n-1, \alpha}$ ($t_t > t_{n-1, \alpha}$)

Review Statistics

Hypothesis testing

For population proportion

	Two-tailed Test	One-tailed Test
Hypothesis	$H_0: p = p_0$ $H_1: p \neq p_0$	$H_0: p > p_0$ ($p < p_0$) $H_1: p \leq p_0$ ($p \geq p_0$)
Test Statistic	$z_t = \dfrac{\hat{p} - p_0}{\sqrt{\dfrac{p_0 q_0}{n}}}$	Where $q_0 = 1 - p_0$
Critical points	$\pm z_{\alpha/2}$	$-z_{\alpha}(z_{\alpha})$
Decision Rules	Reject H_0 if $z_t > z_{\alpha/2}$ or $z_t < -z_{\alpha/2}$	Reject H_0 if $z_t < -z_{\alpha}$ $(z_t > z_{\alpha})$

Review Statistics

Hypothesis testing

For population variance, population is normally distributed

	Two-tailed Test	**One-tailed Test**
Hypothesis	**H_0: $\sigma^2 = \sigma^2_0$** **H_1: $\sigma^2 \neq \sigma^2_0$**	**H_0: $\sigma^2 > \sigma^2_0$ ($\sigma^2 < \sigma^2_0$)** **H_1: $\sigma^2 \leq \sigma^2_0$ ($\sigma^2 \geq \sigma^2_0$)**
Test Statistic	$\chi_t^2 = \dfrac{(n-1)s^2}{\sigma_0^2}$	
Critical Points	$\chi^2_{n-1,1-\alpha/2}$ *or* $\chi^2_{n-1,\alpha/2}$	$\chi^2_{n-1,1-\alpha}$ ($\chi^2_{n-1,\alpha}$)
Decision Rule	Reject Null Hypothesis if $\chi_t^2 < \chi^2_{n-1,1-\alpha/2}$ *or* $\chi_t^2 > \chi^2_{n-1,\alpha/2}$	Reject Null Hypothesis if $\chi_t^2 < \chi^2_{n-1,1-\alpha}$ ($\chi_t^2 > \chi^2_{n-1,\alpha}$)

Review Statistics

Confidence interval

- Population is normal
  - If σ is known

$$\bar{x} \pm z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}$$

  - If σ is not known, small sample size

$$\bar{x} \pm t_{n-1,\alpha/2} \cdot \frac{s}{\sqrt{n}}$$

  - If σ is not known, large sample size

$$\bar{x} \pm z_{\alpha/2} \cdot \frac{s}{\sqrt{n}}$$

- Population is not normal
  - Large sample size

$$\bar{x} \pm z_{\alpha/2} \cdot \frac{s}{\sqrt{n}}$$

(CL Theorem)

  - Small sample size?

Using *Non-parametric techniques*

Random Numbers

Properties of RNs

- Uniformity
- Independence

Generation of pseudo-random numbers

- Linear Congruential Method
- Combined Linear Congruential Method
- Random-Number Streams

Random Numbers

Test for RNs

Test for uniformity (Frequency test)

Kolmogorov-Smirnov test

Arrange $R_{(i)}$ from smallest to largest

Step 1:

$R_{(i)}$	0.05	0.14	0.44	0.81	0.93
i/N	0.20	0.40	0.60	0.80	1.00
$i/N - R_{(i)}$	0.15	0.26	0.16	-	0.07
$R_{(i)} - (i-1)/N$	0.05	-	0.04	0.21	0.13

Step 2:

$D^+ = max\{i/N - R_{(i)}\}$

$D^- = max\{R_{(i)} - (i-1)/N\}$

Step 3: $D = max(D^+, D^-) = 0.26$

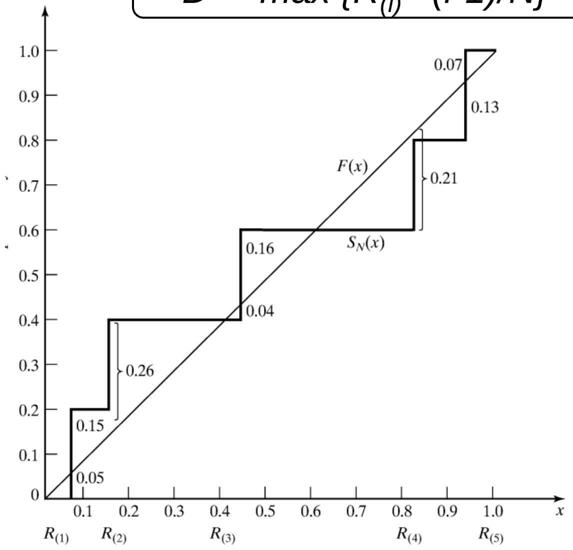
Step 4: For $\alpha = 0.05$,

$D_\alpha = 0.565 > D$

Hence, H_0 is not rejected.

Table A.8 Kolmogorov--Smirnov Critical Values

Degrees of Freedom (N)	$D_{0.10}$	$D_{0.05}$	$D_{0.01}$
1	0.950	0.975	0.995
2	0.776	0.842	0.929
3	0.642	0.708	0.828
4	0.564	0.624	0.733
5	0.510	0.565	0.669
6	0.470	0.521	0.618
7	0.438	0.486	0.577
8	0.411	0.457	0.543
9	0.388	0.432	0.514
10	0.368	0.410	0.490



Random Numbers

Test for RNs

Test for uniformity (Frequency test)

Chi-square test (N>=50)

- Chi-square test uses the sample statistic:

$$\chi_0^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i}$$

n is the # of classes

E_i is the expected # in the i^{th} class

O_i is the observed # in the i^{th} class

  - Approximately the chi-square distribution with *n-1* degrees of freedom (where the critical values are tabulated in Table A.6)
  - For the uniform distribution, E_i, the expected number in each class is:

$$E_i = \frac{N}{n}, \quad \text{where N is the total \# of observation}$$

- Valid only for large samples, e.g., N >= 50

Random Numbers

Test for RNs

Test for independence (Test for autocorrelation)

- Testing the autocorrelation between every l numbers (l is as known as. the lag), starting with the i^{th} number
  - The autocorrelation ρ_{il} between numbers: $R_i, R_{i+l}, R_{i+2l}, R_{i+(M+1)l}$
  - M is the largest integer such that $i+(M+1)l \leq N$

- Hypothesis:

$$H_0: \quad \rho_{il} = 0, \quad \text{if numbers are independent}$$
$$H_1: \quad \rho_{il} \neq 0, \quad \text{if numbers are dependent}$$

- If the values are uncorrelated:
  - For large values of M, the distribution of the estimator of ρ_{il}, denoted is approximately normal.

Random Numbers

Test for RNs

Test for independence (Test for autocorrelation)

- Test statistics is: $$Z_0 = \frac{\hat{\rho}_{il}}{\hat{\sigma}_{\hat{\rho}_{il}}}$$
  - Z_0 is distributed normally with mean = *0* and variance = *1*, and:

$$\hat{\rho}_{il} = \frac{1}{M+1}\left[\sum_{k=0}^{M} R_{i+kl} R_{i+(k+1)l}\right] - 0.25$$

$$\hat{\sigma}_{\rho_{il}} = \frac{\sqrt{13M+7}}{12(M+1)}$$

- If $\rho_{il} > 0$, the subsequence has positive autocorrelation
  - High random numbers tend to be followed by high ones, and vice versa.
- If $\rho_{il} < 0$, the subsequence has negative autocorrelation
  - Low random numbers tend to be followed by high ones, and vice versa.

Random-Variate Generation

Inverse-transform technique

$$X = F^{-1}(R)$$

Continuous distribution: exponential distribution, uniform distribution, triangular distribution, empirical continuous distribution, etc.

Discrete distribution: discrete uniform distribution, geometric distribution, etc.

Random-Variate Generation

Inverse-transform technique: $X = F^{-1}(R)$

Exponential distribution

Pdf:

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Cdf:

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Solve the equation F(X) = R for X in terms of R.

$$\begin{aligned} 1 - e^{-\lambda X} &= R \\ e^{-\lambda X} &= 1 - R \\ -\lambda X &= \ln(1 - R) \\ X &= -\frac{1}{\lambda}\ln(1 - R) \end{aligned}$$

Random-Variate Generation

Inverse-transform technique: $X = F^{-1}(R)$

Uniform distribution

Pdf:

$$f(x) = \begin{cases} \dfrac{1}{b-a}, & a \le x \le b \\ 0, & \text{otherwise} \end{cases}$$

Cdf:

$$F(x) = \begin{cases} 0, & x < a \\ \dfrac{x-a}{b-a}, & a \le x \le b \\ 1, & x > b \end{cases}$$

Solve the equation F(X) = R for X in terms of R.

$$X = a + (b-a)R$$

Random-Variate Generation

Inverse-transform technique: $X = F^{-1}(R)$

Triangular distribution

Consider a random variable X that has pdf:

$$f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2 - x, & 1 < x \le 2 \\ 0, & \text{otherwise} \end{cases}$$

The cdf is then given by:

$$F(x) = \begin{cases} 0, & x \le 0 \\ \dfrac{x^2}{2}, & 0 < x \le 1 \\ 1 - \dfrac{(2-x)^2}{2}, & 1 < x \le 2 \\ 1, & x > 2 \end{cases}$$

Solve the equation F(X) = R for X in terms of R.

$$X = \begin{cases} \sqrt{2R}, & 0 \le R \le \frac{1}{2} \\ 2 - \sqrt{2(1-R)}, & \frac{1}{2} < R \le 1 \end{cases}$$

Random-Variate Generation

Inverse-transform technique: $X = F^{-1}(R)$

Weibull distribution

Pdf:

$$f(x) = \begin{cases} \frac{\beta}{\alpha^{\beta}} x^{\beta-1} e^{-(x/\alpha)^{\beta}}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Cdf:

$$F(X) = 1 - e^{-(x/\alpha)^{\beta}}, x \geq 0$$

Solve the equation F(X) = R for X in terms of R.

$$X = \alpha[-\ln(1 - R)]^{1/\beta}$$

Random-Variate Generation

Inverse-transform technique: $X = F^{-1}(R)$

Empirical discrete distribution

Pdf:

x	$p(x)$	$F(x)$
0	0.50	0.50
1	0.30	0.80
2	0.20	1.00

Cdf:

$$F(x) = \begin{cases} 0 & x < 0 \\ 0.5 & 0 \le x < 1 \\ 0.8 & 1 \le x < 2 \\ 1.0 & 2 \le x \end{cases}$$

Solve the equation F(X) = R for X in terms of R.

$$X = \begin{cases} 0, & R \le 0.5 \\ 1, & 0.5 < R \le 0.8 \\ 2, & 0.8 < R \le 1.0 \end{cases}$$

Random-Variate Generation

Acceptance-rejection technique

Suppose that an analyst needed to devise a method for generating random variates, X, uniformly distributed between 1/4 and 1. One way to proceed would be to follow these steps:

Step 1. Generate a random number R.

Step 2a. If $R \geq 1/4$, accept $X = R$, then go to Step 3.

Step 2b. If $R < 1/4$, reject R, and return to Step 1.

Step 3. If another uniform random variate on [1/4, 1] is needed, repeat the procedure beginning at Step 1. If not, stop

Random-Variate Generation

Acceptance-rejection technique

Poisson Distribution

$$\prod_{i=1}^{n} R_i \geq e^{-\alpha} > \prod_{i=1}^{n+1} R_i$$

Step 1. Set $n = 0$, $P = 1$.

Step 2. Generate a random number $Rn+1$ and replace P by $P \cdot Rn+1$.

Step 3. If $P < e^{-\alpha}$, then accept $N = n$. Otherwise, reject the current n, increase n by one, and return to step 2.

Random-Variate Generation

Acceptance-rejection technique

Poisson Distribution

Generate three Poisson variates with mean $\alpha = 0.2$. First, compute $e^{-\alpha} = e^{-0.2} = 0.8187$. Next, get a sequence of random numbers R from Table A.1 and follow the previously described Steps 1 to 3:

Step 1. Set $n = 0$, $P = 1$.

Step 2. $R_1 = 0.4357$, $P = 1 \cdot R_1 = 0.4357$.

Step 3. Since $P = 0.4357 < e^{-\alpha} = 0.8187$, accept $N = 0$.

Step 1–3. ($R_1 = 0.4146$ leads to $N = 0$.)

Step 1. Set $n = 0$, $P = 1$.

Step 2. $R_1 = 0.8353$, $P = 1 \cdot R_1 = 0.8353$.

Step 3. Since $P \geq e^{-\alpha}$, reject $n = 0$ and return to Step 2 with $n = 1$.

Step 2. $R_2 = 0.9952$, $P = R_1 R_2 = 0.8313$.

Step 3. Since $P \geq e^{-\alpha}$, reject $n = 1$ and return to Step 2 with $n = 2$.

Step 2. $R_3 = 0.8004$, $P = R_1 R_2 R_3 = 0.6654$.

Step 3. Since $P < e^{-\alpha}$, accept $N = 2$.

Random-Variate Generation

Special properties

Some random-variate generation methods are based on features of a particular family of probability distributions. For example:

- Direct transformation for normal and lognormal distributions
- Convolution method
- Beta distribution from gamma distribution

Best luck!

Spring Semester, 2022 – 2023

Midterm Review