Simulation Models in Industrial Engineering

Spring Semester, 2022 – 2023

Final Review

Outline

Lecture 7 — Input Modeling

Lecture 8 — Verification and Validation

Lecture 9 — Absolute Output Analysis

Lecture 10 — Relative Output Analysis

Content Review

Lecture 7 — Input Modeling

Lecture 8 — Verification and Validation

Lecture 9 — Absolute Output Analysis

Lecture 10 — Relative Output Analysis

Input Modeling | Summary

Development of a useful model of input data:

Step 1: Data collection

Step 2: Identifying the distribution with data

- Histogram
- Selecting the family of distributions
- Quantile-Quantile (Q-Q) plot

Step 3: Parameter estimation

- Sample mean and sample variance
- Suggested estimators

Step 4: Goodness-of-fit tests

- Graphical approach: Q-Q plot
- Statistical test: Chi-square test, Kolmogorov–Smirnov test (K-S test)

Input Modeling | Data collection

Development of a useful model of input data:

Step 1: Data collection

A few examples…

- Interarrival time: clock times are recorded on entity arrivals to form the sequence of entity interarrival time
- Processing time: the time it takes to finish a job
- Time to failure: the time interval between system failures that result in a stoppage of system operation
- Downtime/ Repair time: the time from stoppage to the time system operation is resumed

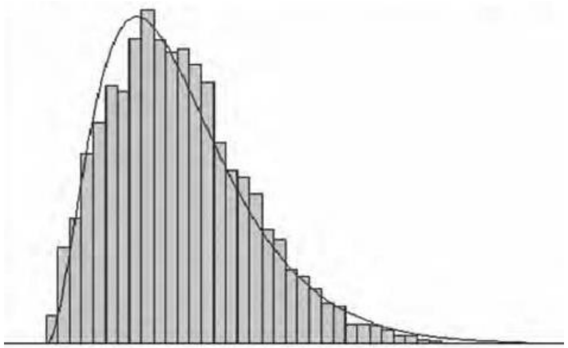
Q1, in-class quiz 1&2

Input Modeling | Identifying distribution

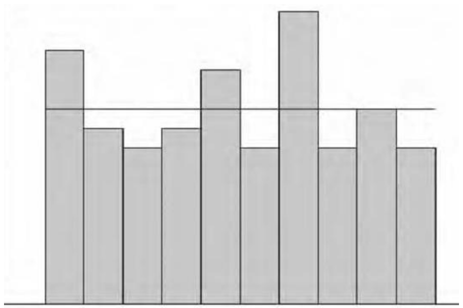
Development of a useful model of input data:

Step 2: Identifying the distribution with data

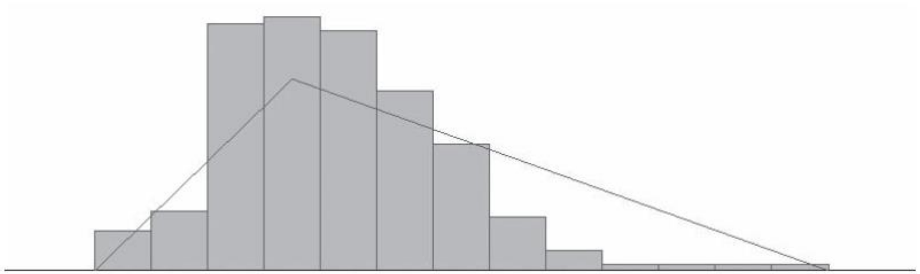
👉 Histogram



Normal distribution



Uniform distribution



Triangular distribution

Q5a, in-class quiz 1

Input Modeling | Identifying distribution

Development of a useful model of input data:

Step 2: Identifying the distribution with data

👉 Selecting the family of distribution

Binomial: Models the number of successes in n trials, when the trials are independent with common success probability p; for example, the number of defective computer chips found in a lot of n chips.

Negative Binomial (includes the geometric distribution): Models the number of trials required to achieve k successes; for example, the number of computer chips that we must inspect to find 4 defective chips.

Poisson: Models the number of independent events that occur in a fixed amount of time or space; for example, the number of customers that arrive to a store during 1 hour, or the number of defects found in 30 square meters of sheet metal.

Q2, in-class quiz 1&2

Input Modeling | Identifying distribution

Development of a useful model of input data:

Step 2: Identifying the distribution with data

👉 Selecting the family of distribution

Normal: Models the distribution of a process that can be thought of as the sum of a number of component processes; for example, a time to assemble a product that is the sum of the times required for each assembly operation. Notice that the normal distribution admits negative values, which could be impossible for process times.

Lognormal: Models the distribution of a process that can be thought of as the product of (meaning to multiply together) a number of component processes; for example, the rate of return on an investment, when interest is compounded, is the product of the returns for a number of periods.

Q2, in-class quiz 1&2

Input Modeling | Identifying distribution

Development of a useful model of input data:

Step 2: Identifying the distribution with data

Selecting the family of distribution

Exponential: Models the time between independent events, or a process time that is memoryless (knowing how much time has passed gives no information about how much additional time will pass before the process is complete); for example, the times between the arrivals from a large population of potential customers who act independently of one another. The exponential is a highly variable distribution; it is sometimes overused, because it often leads to mathematically tractable models. Recall that, if the time between events is exponentially distributed, then the number of events in a fixed period of time is Poisson.

Gamma: An extremely flexible distribution used to model nonnegative random variables. The gamma can be shifted away from 0 by adding a constant.

Q2, in-class quiz 1&2

Input Modeling | Identifying distribution

Development of a useful model of input data:

Step 2: Identifying the distribution with data

- Selecting the family of distribution

Beta: An extremely flexible distribution used to model bounded (fixed upper and lower limits) random variables. The beta can be shifted away from 0 by adding a constant and can be given a range larger than [0, 1] by multiplying by a constant.

Erlang: Models processes that can be viewed as the sum of several exponentially distributed processes; for example, a computer network fails when a computer and two backup computers fail, and each has a time to failure that is exponentially distributed. The Erlang is a special case of the gamma.

Weibull: Models the time to failure for components; for example, the time to failure for a disk drive. The exponential is a special case of the Weibull.

Q2, in-class quiz 1&2

Input Modeling | Identifying distribution

Development of a useful model of input data:

Step 2: Identifying the distribution with data

Selecting the family of distribution

Discrete or Continuous Uniform: Models complete uncertainty: All outcomes are equally likely. This distribution often is used inappropriately when there are no data.

Triangular: Models a process for which only the minimum, most likely, and maximum values of the distribution are known; for example, the minimum, most likely, and maximum time required to test a product. This model is often a marked improvement over a uniform distribution.

Empirical: Resamples from the actual data collected; often used when no theoretical distribution seems appropriate.

Q2, in-class quiz 1&2

Input Modeling | Identifying distribution

Development of a useful model of input data:

Step 2: Identifying the distribution with data

Q-Q plot

The idea behind Q-Q plot….

If X is a random variable with cdf F, then the q-quantile of X is that value γ such that $F(\gamma) = P(X \le \gamma) = q$, for $0 < q < 1$. When F has an inverse, we write $\gamma = F^{-1}(q)$.

Now let $\{x_i, i = 1, 2, \ldots, n\}$ be a sample of data from X. Order the observations from the smallest to the largest, and denote these as $\{y_j, j = 1, 2, \ldots n\}$, where $y_1 \le y_2 \le \cdots \le y_n$. Let j denote the ranking or order number. Therefore, $j = 1$ for the smallest and $j = n$ for the largest. The q-q plot is based on the fact that y_j is an estimate of the $(j - 1/2)/n$ quantile of X. In other words,

$$y_j \text{ is approximately } F^{-1}\left(\frac{j - \frac{1}{2}}{n}\right)$$

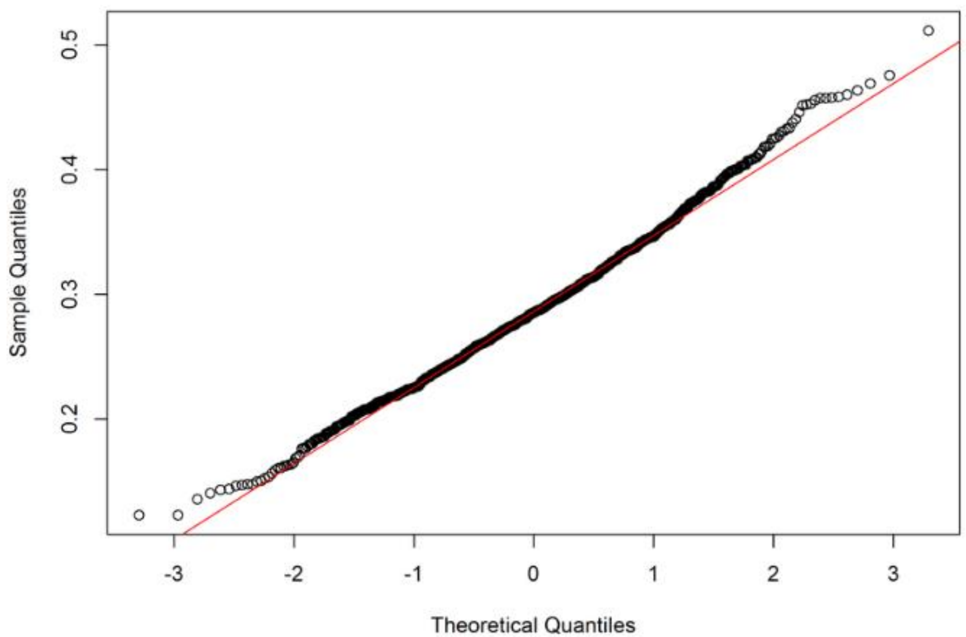
Q5b, in-class quiz 1

Input Modeling | Identifying distribution

Development of a useful model of input data:

Step 2: Identifying the distribution with data

Q-Q plot



If F is a member of an appropriate family of distributions, then its Q-Q plot will be approximately a straight line.

If F is from an appropriate family of distributions and also has appropriate parameter values, then the line will have slope 1.

On the other hand, if the assumed distribution is inappropriate, the points will deviate from a straight line, usually in a systematic manner.

The decision about whether to reject some hypothesized model is subjective.

Q5b, in-class quiz 1

Input Modeling | Identifying distribution

Development of a useful model of input data:

Step 3: Parameter estimation

- Sample mean and sample variance

When discrete or continuous raw data are available:

$$\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n} \qquad (1)$$

$$S^2 = \frac{\sum_{i=1}^{n} X_i^2 - n\bar{X}^2}{n-1} \qquad (2)$$

Q4, in-class quiz 2

Input Modeling | Identifying distribution

Development of a useful model of input data:

Step 3: Parameter estimation

Sample mean and sample variance

When the data are discrete and have been grouped in a frequency distribution:

$$\bar{X} = \frac{\sum_{j=1}^{k} f_j X_j}{n} \tag{3}$$

$$S^2 = \frac{\sum_{j=1}^{k} f_j X_j^2 - n\bar{X}^2}{n-1} \tag{4}$$

Q4, in-class quiz 1

Input Modeling | Identifying distribution

Development of a useful model of input data:

Step 3: Parameter estimation

👉 Sample mean and sample variance

When the data are discrete or continuous and have been placed in class intervals and their raw data are unavailable **!**

$$\bar{X} \doteq \frac{\sum_{j=1}^{c} f_j m_j}{n} \tag{5}$$

$$S^2 \doteq \frac{\sum_{j=1}^{c} f_j m_j^2 - n\bar{X}^2}{n-1} \tag{6}$$

where f_j is the observed frequency in the jth class interval, m_j is the midpoint of the jth interval, and c is the number of class intervals.

Now consider the following example… bonus question!

Input Modeling | Identifying distribution

Development of a useful model of input data:

Step 3: Parameter estimation

Bonus question!

Calculate the sample mean and variance, given:

$$\bar{X} \doteq \frac{\sum_{j=1}^{c} f_j m_j}{n} \tag{5}$$

$$S^2 \doteq \frac{\sum_{j=1}^{c} f_j m_j^2 - n\bar{X}^2}{n-1} \tag{6}$$

Component Life (days)	*Frequency*
$0 \le x_j < 3$	23
$3 \le x_j < 6$	10
$6 \le x_j < 9$	5
$9 \le x_j < 12$	1
$12 \le x_j < 15$	1
$15 \le x_j < 18$	2
$18 \le x_j < 21$	0
$21 \le x_j < 24$	1
$24 \le x_j < 27$	1
$27 \le x_j < 30$	0
$30 \le x_j < 33$	1
$33 \le x_j < 36$	1
⋮	⋮
$42 \le x_j < 45$	1
⋮	⋮
$57 \le x_j < 60$	1
⋮	⋮
$78 \le x_j < 81$	1
⋮	⋮
$144 \le x_j < 147$	1

Input Modeling | Identifying distribution

Development of a useful model of input data:

Step 3: Parameter estimation

- Sample mean and sample variance

$$\bar{X} \doteq \frac{\sum_{j=1}^{c} f_j m_j}{n} \qquad (5)$$

$$\bar{X} \doteq \frac{614}{50} = 12.28$$

$$S^2 \doteq \frac{\sum_{j=1}^{c} f_j m_j^2 - n\bar{X}^2}{n-1} \qquad (6)$$

$$S^2 \doteq \frac{37{,}226.5 - 50(12.28)^2}{49} = 605.849$$

Component Life (*days*)	*Frequency*
$0 \le x_j < 3$	23
$3 \le x_j < 6$	10
$6 \le x_j < 9$	5
$9 \le x_j < 12$	1
$12 \le x_j < 15$	1
$15 \le x_j < 18$	2
$18 \le x_j < 21$	0
$21 \le x_j < 24$	1
$24 \le x_j < 27$	1
$27 \le x_j < 30$	0
$30 \le x_j < 33$	1
$33 \le x_j < 36$	1
⋮	⋮
$42 \le x_j < 45$	1
⋮	⋮
$57 \le x_j < 60$	1
⋮	⋮
$78 \le x_j < 81$	1
⋮	⋮
$144 \le x_j < 147$	1

Input Modeling | Identifying distribution

Development of a useful model of input data:

Step 3: Parameter estimation

- Suggested estimators

Table 3 Suggested Estimators for Distributions Often Used in Simulation

Distribution	*Parameter(s)*	*Suggested Estimator(s)*
Poisson	α	$\widehat{\alpha} = \bar{X}$
Exponential	λ	$\widehat{\lambda} = \frac{1}{\bar{X}}$
Gamma	β, θ	$\widehat{\beta}$ (see Table A.9) $\widehat{\theta} = \frac{1}{\bar{X}}$
Normal	μ, σ^2	$\widehat{\mu} = \bar{X}$ $\widehat{\sigma}^2 = S^2$ (unbiased)
Lognormal	μ, σ^2	$\widehat{\mu} = \bar{X}$ (after taking ln of the data) $\widehat{\sigma}^2 = S^2$ (after taking ln of the data)
Weibull with $\nu = 0$	α, β	$\widehat{\beta}_0 = \frac{\bar{X}}{S}$ $\widehat{\beta}_j = \widehat{\beta}_{j-1} - \frac{f(\widehat{\beta}_{j-1})}{f'(\widehat{\beta}_{j-1})}$ See Equations (11) and (14) for $f(\widehat{\beta})$ and $f'(\widehat{\beta})$ Iterate until convergence $\widehat{\alpha} = \left(\frac{1}{n}\sum_{i=1}^{n} X_i^{\widehat{\beta}}\right)^{1/\widehat{\beta}}$
Beta	β_1, β_2	$\Psi(\widehat{\beta}_1) + \Psi(\widehat{\beta}_1 - \widehat{\beta}_2) = \ln(G_1)$ $\Psi(\widehat{\beta}_2) + \Psi(\widehat{\beta}_1 - \widehat{\beta}_2) = \ln(G_2)$ where Ψ is the digamma function, $G_1 = \left(\prod_{i=1}^{n} X_i\right)^{1/n}$ and $G_2 = \left(\prod_{i=1}^{n} (1 - X_i)\right)^{1/n}$

Q4, in-class quiz 1&2

Input Modeling | Identifying distribution

Development of a useful model of input data:

Step 4: Goodness-of-fit test

👉 Statistical test: Chi-square test

Hypothesis	H_0: The random variable, X, conforms to the distributional assumption with the parameter(s) given by the parameter estimate(s). H_1: The random variable X does not conform.
Test statistics	$$X_0^2 = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i}$$ O_i: the observed frequency in the i^{th} class interval E_i: the expected frequency in that class interval, $E_i = np_i$ where p_i is the theoretical, hypothesized probability associated with the i^{th} class interval.
Critical values	$X^2_{\alpha,k-s-1}$ (Table A.6) $k - s - 1$: degrees of freedom, where s *is* the number of parameters of the hypothesized distribution estimated by the sample statistics
Conclusion	The null hypothesis H_0 is rejected if $X_0^2 > X^2_{\alpha,k-s-1}$

Q5c, in-class quiz 1 & Q5, in-class quiz 2

Input Modeling | Identifying distribution

Development of a useful model of input data:

Step 4: Goodness-of-fit test

Statistical test: Chi-square test with equal probabilities

If a continuous distributional assumption is being tested, class intervals that are equal in *probability* rather than equal in width of interval should be used.*** (The endpoints of the class intervals must be found.)

However, if using equal probabilities, then $p_i = \frac{1}{k}$. It is recommended that:

$$E_i = np_i \geq 5 \quad \therefore \quad n\frac{1}{k} \geq 5 \quad \therefore \quad k \leq \frac{n}{5}$$

If the assumed distribution is normal, exponential, or Weibull, the method described is straightforward.

If the assumed distribution is gamma (but not Erlang) or certain other distributions, then the computation of endpoints for class intervals is complex and could require numerical integration of the density function. Statistical-analysis software is very helpful in such cases.

**** This has been recommended by a number of authors [Mann and Wald, 1942; Gumbel, 1943; Law, 2007; Stuart, Ord, and Arnold, 1998].*

Input Modeling | Identifying distribution

Development of a useful model of input data:

Step 4: Goodness-of-fit test

👉 Statistical test: K-S test

Hypothesis	H_0: The random variable, X, conforms to the distributional assumption with the parameter(s) given by the parameter estimate(s). H_1: The random variable X does not conform.
Test statistics	$D = \max\|F(x) - S_N(x)\| = \max(D^+, D^-)$ $F(x)$: the continuous cdf of the distribution $S_N(x)$: the empirical cdf of the sample of N observations $D^+ = max_{1\le i\le N}\left\{\frac{i}{N} - R_i\right\}, D^- = max_{1\le i\le N}\left\{R_i - \frac{i-1}{N}\right\}$
Critical values	D_α (Table A.8) with sample size N
Conclusion	The null hypothesis $H_0$is rejected if $D > D_\alpha$

Input Modeling | Identifying distribution

Development of a useful model of input data:

Step 4: Goodness-of-fit test

Graphical approach: Q-Q plot

99.79	99.56	100.17	100.33
100.26	100.41	99.98	99.83
100.23	100.27	100.02	100.47
99.55	99.62	99.65	99.82
99.96	99.90	100.06	99.85

Bonus question!

Do a goodness of fit test for the given data using Q-Q plot approach.

Using Q-Q plot to test if given data is normally distributed, given mean is 99.99 and std. dev. Is 0.2832.

Given the **cumulative distribution function** is given by:

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{x^2/2} dx, -\infty < z < \infty$$

Hint: Using Casio

Input Modeling | Identifying distribution

Development of a useful model of input data:

Step 4: Goodness-of-fit test

👉 Graphical approach: Q-Q plot

What to remember when doing goodness-of-fit test using Q-Q plot?
Cumulative distribution functions!

Here are other widely used cdf., take note!

UNIF distribution:

$$q = F_X(x) = \begin{cases} 0, & \text{if } x < a \\ \frac{x-a}{b-a}, & \text{if } a \le x \le b \\ 1, & \text{if } x > b. \end{cases}$$

$$x = F_X^{-1}(q) = q(b-a) + a$$

EXPO distribution:

$$q = F_X(x) = 1 - e^{-\lambda x}, x \ge 0$$

$$x = F_X^{-1}(q) = -\frac{1}{\lambda}\ln(1-q)$$

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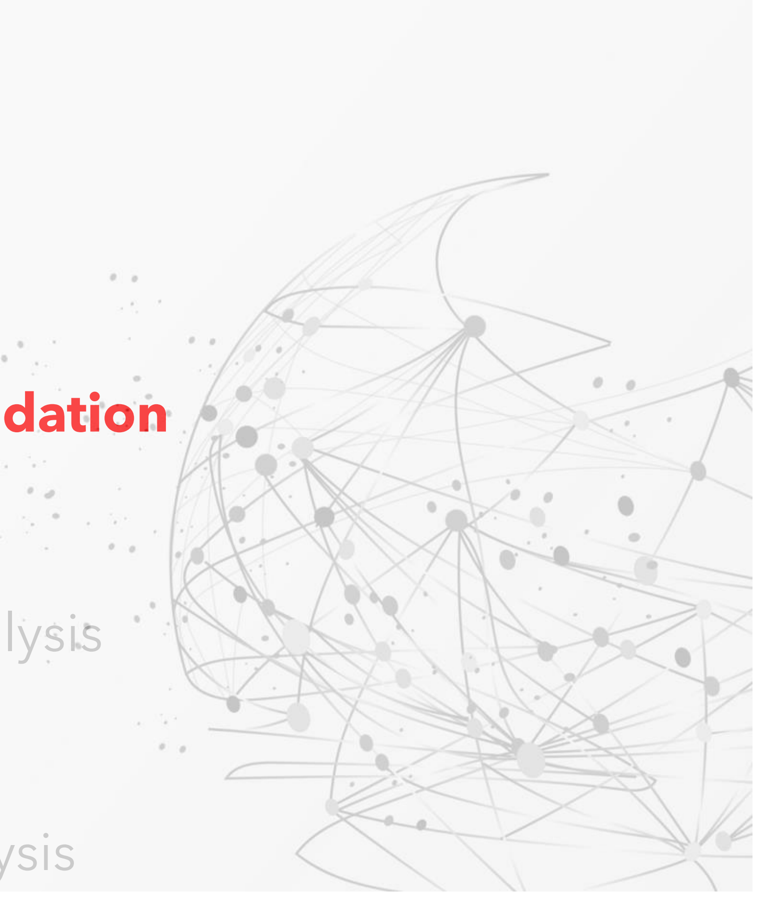

Verification & Validation | Summary

Model building, Verification, and Validation

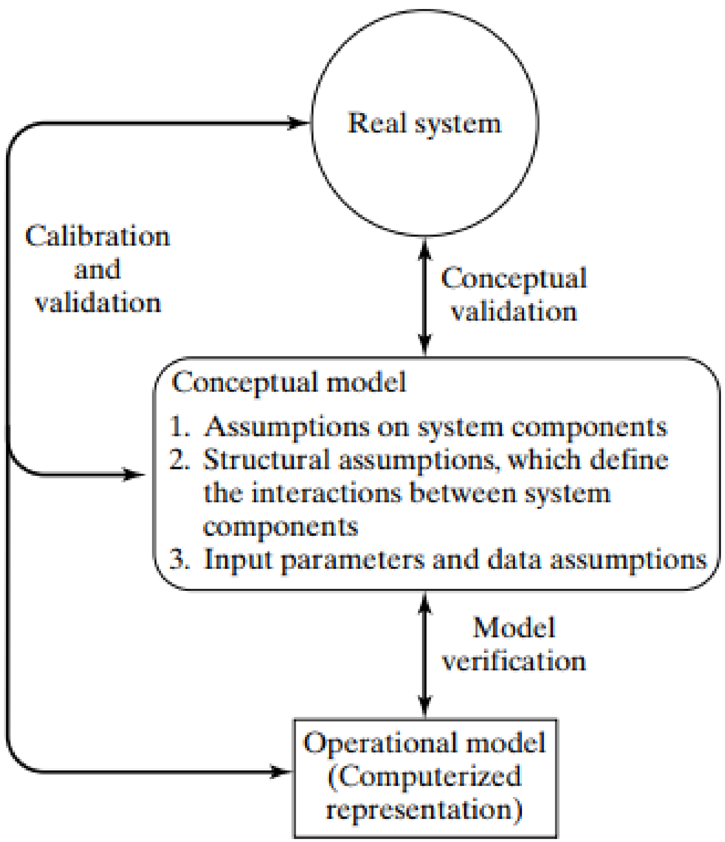

Figure 1 Model building, verification, and validation.

Verification

Is concerned with building the model right

Compares the conceptual model to the computer representation that implements that conception

Validation

Is concerned with building the correct model

Confirms that a model is an accurate representation of the real system

Verification & Validation | Summary

Validating Input-Output transformations

Hypothesis	H_0: $\mathrm{E(Y)} =, \geq, \leq\ \mu_0$ (Model output is consistent with system behavior.) H_1: $\mathrm{E(Y)} \neq, <, >\ \mu_0$ (Model output is inconsistent with system behavior.)
Test statistics	$t_0 = \dfrac{\bar{Y} - \mu_0}{\frac{S}{\sqrt{n}}}$ $\bar{Y} = \dfrac{1}{n}\sum_{i=1}^{n} Y_i\, , S^2 = \dfrac{\sum_{i=1}^{n}(Y_i - \bar{Y})^2}{n-1}$
Critical values	$t_{\frac{\alpha}{2},n-1}$ (Table A.5)
Conclusion	Two-sided test: Reject H_0 if $\lvert t_0 \rvert > t_{\frac{\alpha}{2},n-1}$ One-sided test: Reject H_0 if $t_0 > t_{\alpha,n-1}$ or $t_0 < t_{\alpha,n-1}$
Power of the test	$1 - \beta(\delta), \delta = \dfrac{\lvert E(Y) - \mu_0 \rvert}{\sigma}$ (Table A.11)

Ex3, Prob1&2&3, HW4

Verification & Validation | Summary

Types of error in model validation

Table 4 Types of Error in Model Validation

Statistical Terminology	*Modeling Terminology*	*Associated Risk*
Type I: rejecting H_0 when H_0 is true	Rejecting a valid model	α
Type II: failure to reject H_0 when H_1 is true	Failure to reject an invalid model	β

1. Suppose the confidence interval does not contain μ_0. [See Figure 6(a).]
   (a) If the best-case error is $> \varepsilon$, then the difference in performance is large enough, even in the best case, to indicate that we need to refine the simulation model.
   (b) If the worst-case error is $\leq \varepsilon$, then we can accept the simulation model as close enough to be considered valid.
   (c) If the best-case error is $\leq \varepsilon$, but the worst-case error is $> \varepsilon$, then additional simulation replications are necessary to shrink the confidence interval until a conclusion can be reached.
2. Suppose the confidence interval does contain μ_0. [See Figure 6(b).]
   (a) If either the best-case or worst-case error is $> \varepsilon$, then additional simulation replications are necessary to shrink the confidence interval until a conclusion can be reached.
   (b) If the worst-case error is $\leq \varepsilon$, then we can accept the simulation model as close enough to be considered valid.

Ex3, Prob1&2&3, HW4

Verification & Validation | Summary

Types of error in model validation

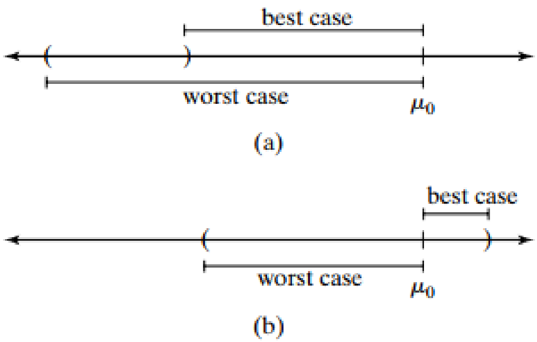

Figure 6 Validation of the input-output transformation (a) when the known value falls outside, and (b) when the known value falls inside, the confidence interval.

1. Suppose the confidence interval does not contain μ_0. [See Figure 6(a).]
   (a) If the best-case error is $> \varepsilon$, then the difference in performance is large enough, even in the best case, to indicate that we need to refine the simulation model.
   (b) If the worst-case error is $\leq \varepsilon$, then we can accept the simulation model as close enough to be considered valid.
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Ex3, Prob1&2&3, HW4

Content Review

Lecture 7 Input Modeling

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Lecture 10 Relative Output Analysis

Absolute Performance | Summary

Purposes:

To **predict** the performance of a system

Q1, Practice Exercises

Types of Simulations with Respect to Output Analysis:

Terminating simulation: One that runs for some duration of time T_E, where E is a specified event (or set of events) that stops the simulation. Such a simulated system opens at time 0 under well-specified initial conditions and closes at the stopping time T_E.

Steady-state simulation: One whose objective is to study long-run, or steady-state, behavior of a nonterminating system. It starts at time 0 under initial conditions defined by the analyst and runs for some analyst-specified period of time T_E. Usually, the analyst wants to study steady-state properties of the system, properties that are *not* influenced by the initial conditions of the model.

Q3, Practice Exercises

Absolute Performance | Point estimation

- Estimation of absolute performance

Point estimation: Given the data $\{Y_1, \ldots, Y_n\}$, its point estimator is defined by:

$$\bar{Y} = \sum_{i=1}^{n} Y_i \,/n$$

The point estimator $\bar{Y}$ is said to be unbiased for θ if $E(\bar{Y}) = \theta$.

But $\bar{Y}$ is not θ, it is an estimate, based on a sample, and it has error:

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \bar{Y})^2$$

What does this computation assume about the data distribution?

Absolute Performance | C.I. & P.I.

Estimation of absolute performance

Confidence interval estimation (measure of error):

$$\bar{Y} \pm t_{\frac{\alpha}{2},n-1} \frac{S}{\sqrt{n}}$$

(See table A.5 for $t_{\frac{\alpha}{2},n-1}$)

Prediction interval estimation (measure of risk):

$$\bar{Y} \pm t_{\frac{\alpha}{2},n-1} S \sqrt{1 + \frac{1}{n}}$$

We can simulate away error by making more and more replications, but we can never simulate away risk, which is an inherent part of the system. We can, however, do a better job of evaluating risk by making more replications.

Q2, Practice Exercises

Absolute Performance | C.I. Halfwidth

- Estimation of absolute performance

Confidence interval half-width:

$$H = t_{\frac{\alpha}{2},n-1} \frac{S}{\sqrt{n}} \quad \text{where } \frac{S}{\sqrt{n}} \text{ is the standard error}$$

Q4, Practice Exercises

Absolute Performance | C.I. with precision

- Estimation of absolute performance

Confidence interval with specified precision:

Assume that an initial sample of size R_0 replications has been observed—that is, the simulation analyst initially makes R_0 independent replications. We must have $R_0 \geq 2$, with 10 or more being desirable. The R_0 replications will be used to obtain an initial estimate S_0^2 of the population variance σ^2. To meet the half-length criterion, a sample size R must be chosen such that $R \geq R_0$ and

$$H = t_{\alpha/2,R-1} \frac{S_0}{\sqrt{R}} \leq \epsilon \tag{14}$$

Solving for R in Inequality (14) shows that R is the smallest integer satisfying $R \geq R_0$ and

$$R \geq \left(\frac{t_{\alpha/2,R-1} S_0}{\epsilon} \right)^2 \tag{15}$$

An initial estimate for R is given by

$$R \geq \left(\frac{z_{\alpha/2} S_0}{\epsilon} \right)^2 \tag{16}$$

Q5, Practice Exercises

Content Review

Lecture 7 Input Modeling

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Relative Output Analysis | Summary

Purposes:

To compare the performance of two or more alternative system designs.

Q1, Practice Exercises

Compare systems:

C.I. bounds, with probability $1-\alpha$, the true difference $\theta_1 - \theta_2$ within the range:

$$\boxed{\bar{Y}_{.1} - \bar{Y}_{.2} - t_{\alpha/2,\nu}\text{s.e.}(\bar{Y}_{.1} - \bar{Y}_{.2}) \le \theta_1 - \theta_2 \le \bar{Y}_{.1} - \bar{Y}_{.2} + t_{\alpha/2,\nu}\text{s.e.}(\bar{Y}_{.1} - \bar{Y}_{.2})}$$

Independent sampling: Different and independent random number streams will be used to simulate the two systems. This implies that all the observations of simulated system 1 are statistically independent of all the observations of simulated system 2.

Common Random Numbers (CRN): For each replication, the same random numbers are used to simulate both systems. Thus, for each replication, the two estimates are no longer independent, but rather are correlated.

Relative Performance | Independent Sampling

Independent sampling:

Point estimate:

$$\bar{Y}_1 - \bar{Y}_2$$

Standard error of the point estimate:

$$\text{s.e.}(\bar{Y}_1 - \bar{Y}_2) = \sqrt{\frac{S_1^2}{R_1} + \frac{S_2^2}{R_2}}$$

Degree of freedom:

$$v = \frac{(S_1^2/R_1 + S_2^2/R_2)^2}{[(S_1^2/R_1)^2/(R_1 - 1)] + [(S_2^2/R_2)^2/(R_2 - 1)]}$$

A minimum number of replications $R_1 \geq 6$ and $R_2 \geq 6$ is recommended for this procedure.

Ex2, Pt1, HW2

Relative Performance | CRN

CRN

Point estimate:

$$\bar{D} = \frac{1}{R}\sum_{r=1}^{R} D_r \quad where \quad D_r = Y_{r1} - Y_{r2}$$

Variance of the point estimate:

$$S_D^2 = \frac{1}{R-1}\sum_{r=1}^{R}(D_r - \bar{D})^2$$

Standard error of the point estimate:

$$\text{s.e.}(\bar{D}) = s.e.(\bar{Y}_1 - \bar{Y}_2) = \frac{S_D}{\sqrt{R}}$$

Degree of freedom:

$$v = R - 1$$

Q6, Practice Exercises

Relative Performance | CRN

C.I. with specified precision

We want the error in our estimate of $\theta_1 - \theta_2$ to be less than $\pm\epsilon$.

Therefore, our goal is to find a number of replication R such that:

$$H = t_{\frac{\alpha}{2},v} s.e.(\bar{Y}_1 - \bar{Y}_2) \leq \epsilon$$

Thus:

$$H = \frac{\mathrm{t}_{\frac{\alpha}{2},\mathrm{R}-2}\mathrm{S}_\mathrm{D}}{\sqrt{R}} \leq \epsilon$$

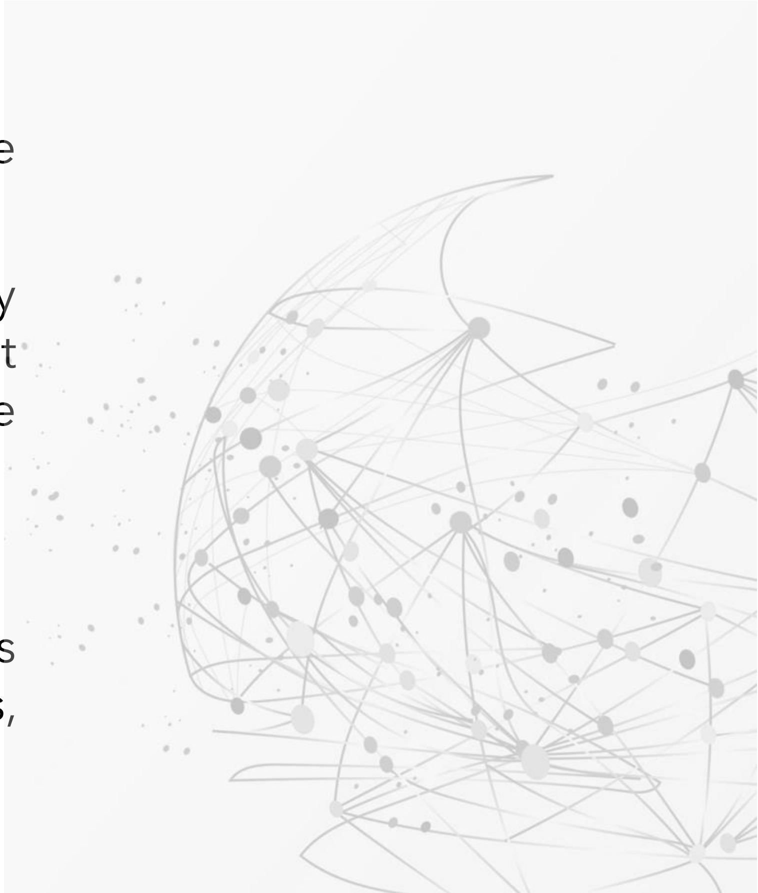
We can approximate this by finding the smallest integer R satisfying $R > R_0$, and:

$$R \geq \left(\frac{z_{\frac{\alpha}{2}} S_D}{\epsilon}\right)^2$$

Q6, Practice Exercises

Final notes!

- Don’t forget to bring reference table and use the right one for your test!
- Be careful and don’t miss out on any questions! Many of you have left at least one question unanswered in the midterm exam.
- When provide examples, be specific.
- For hypothesis testing, remember parts of the test: hypotheses, test statistics, critical values, C.I., conclusion.



It's the end.
Best luck on the exam!

Spring Semester, 2022 – 2023

Course Finished